TWO COUNTER EXAMPLES FOR CONNECTEDNESS

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ABSTRACT

Two examples are presented to show that a k-row by b-column design with v treatments may be row and treatment connected, row and column connected, and treatment and column connected, but may not be a connected design.

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It has been stated that a k-row by b-column design with v treatments replicated r times each, is connected <u>if</u> the rows and treatments, the columns and treatments, and the rows and columns are each connected separately, i.e. there is pairwise connectedness among the three entities. Other than the trivial statements of connectedness concerning the rank of X'X for the $1 + k + b + v$ normal equations written in the form $X'X\underset{\sim}{\beta} = EX'Y$, where the rank must be $1 + k - 1 + b - 1 + v - 1$ if the design is connected, necessary and sufficient conditions for connectedness in k-row by b-column designs need to be developed. The purpose of this note is to present two examples which are row-treatment connected, are column-treatment connected, and are row-column connected but are not row-column-treatment connected.

Let $v = 6$, $r = 2$, $b = 3$, $k = 4$, $m_{r_1} = 3 = m_{c_1}$, $m_{r_2} = 2 = m_{c_2}$, $\lambda_{r_1} = 0$, $\lambda_{r_2} = 1$, $\lambda_{c_1} = 2$, $\lambda_{c_2} = 1$ where these letters have the usual definitions in group-divisible incomplete block designs. The design is

Row	Column 1	Column 2	Column 3
1	1	2	3
2	5	1	4
3	4	6	2
4	3	5	6

Row Association

1	6
2	5
3	4

Column Association

1	5
2	6
3	4

$$N_r = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad N_c = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$N_r N_r' = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

which may be written as $2I_{2\times2} \times I_{3\times3} - J_{2\times2} \times I_{3\times3} + J_{6\times6}$ for treatment order 123654, for I equal to the identity matrix, J equal to a matrix whose elements are unity, and for $\times$ denoting the Kronecker product. Likewise,

$$N_c N_c' = \begin{bmatrix} 2 & 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 \end{bmatrix}$$

which may be written as $I_{3\times3} \times J_{2\times2} + J_{6\times6}$ for the treatment order 162534 and for the other symbols defined as above.

Note that the design is row and treatment connected since $2I-\frac{1}{3}N_rN_r'+\frac{1}{3}J$ has an inverse as follows:

$$\begin{bmatrix} 5/3 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 5/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 5/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 5/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 5/3 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 5/3 \end{bmatrix}^{-1} = \frac{3}{24}\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & -1 \\ 0 & 5 & 0 & 0 & -1 & 0 \\ 0 & 0 & 5 & -1 & 0 & 0 \\ 0 & 0 & -1 & 5 & 0 & 0 \\ 0 & -1 & 0 & 0 & 5 & 0 \\ -1 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Also, note that the design is column and treatment connected since $2I-\frac{1}{4}N_cN_c'$ $+\frac{1}{4}J$ has an inverse as follows:

$$\begin{bmatrix} 7/4 & 0 & 0 & 0 & -1/4 & 0 \\ 0 & 7/4 & 0 & 0 & 0 & -1/4 \\ 0 & 0 & 7/4 & -1/4 & 0 & 0 \\ 0 & 0 & -1/4 & 7/4 & 0 & 0 \\ -1/4 & 0 & 0 & 0 & 7/4 & 0 \\ 0 & -1/4 & 0 & 0 & 0 & 7/4 \end{bmatrix}^{-1} = \frac{4}{48}\begin{bmatrix} 7 & 0 & 0 & 0 & 1 & 0 \\ 0 & 7 & 0 & 0 & 0 & 1 \\ 0 & 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 & 0 & 0 \\ 1 & 0 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 & 7 \end{bmatrix}$$

The design is obviously row and column connected since rows are orthogonal to columns.

The matrix $2I-\frac{1}{3}N_rN_r'-\frac{1}{4}N_cN_c'$ has rank 4 instead of 5 which is required to obtain row, column, and treatment connectedness. Thus, $2I-\frac{1}{3}N_rN_r'-\frac{1}{4}N_cN_c'+\frac{7}{12}J$ is equal to:

$$\frac{1}{12}\begin{bmatrix} 5 & 0 & 0 & 0 & -3 & 4 \\ 0 & 5 & 0 & 0 & 4 & -3 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ -3 & 4 & 0 & 0 & 5 & 0 \\ 4 & -3 & 0 & 0 & 0 & 5 \end{bmatrix}$$

We may rewrite the above matrix as follows:

$$\begin{bmatrix} 5 & 0 & 0 & 0 & -3 & 4 \\ 0 & 5 & 0 & 0 & 4 & -3 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 24/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 24/5 \\ 0 & 0 & 0 & 0 & 0 & 18/5 \end{bmatrix} = \begin{bmatrix} \text{row } 1=Q_1 \\ \text{row } 2=Q_2 \\ \text{row } 3=Q_3 \\ (\text{row } 4-Q_4)-Q_3/5 \\ (\text{row } 5=Q_5)+3Q_1/5-4Q_2/5 \\ (\text{row } 6=Q_6)-4Q_1/5-3Q_2/5 \end{bmatrix}$$

Hence, the above is a counter example to the claim that if a design is pairwise connected among the three entities -- rows, columns, and treatments -- that it is overall connected.

A second counter example is the following 3 x 3 latin square with two plots missing as follows:

	Column		
Row	1	2	3
1	-	B	C
2	B	-	A
3	C	A	B

Rows and treatments are connected, columns and treatments are connected, and rows and columns are connected but the design is not row, column, and treatment connected; the normal equations are

$$\mu: \quad 7\mu+2\rho_1+2\rho_2+3\rho_3+2\gamma_1+2\gamma_2+3\gamma_3+2\tau_A+3\tau_B+2\tau_C = Y_{...}$$

$$\rho_h: \quad n_{h..}(\mu+\rho_h)+\Sigma n_{hi.}\gamma_i+\Sigma n_{h.j}\tau_j = Y_{h..}$$

$$\gamma_i: \quad n_{.i.}(\mu+\gamma_i)+\Sigma n_{hi.}\rho_h+\Sigma n_{.ij}\tau_j = Y_{.i.}$$

$$\tau_j: \quad n_{..j}(\mu+\tau_j)+\Sigma n_{h.j}\rho_h+\Sigma n_{.ij}\gamma_i = Y_{..j}$$

Or in matrix form the above may be written as:

$$\begin{bmatrix} 7 & 2 & 2 & 3 & 2 & 2 & 3 & 2 & 3 & 2 \\ 2 & 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 & 3 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & 0 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \mu \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \tau_A \\ \tau_B \\ \tau_C \end{bmatrix} = \begin{bmatrix} Y_{...} \\ Y_{1..} \\ Y_{2..} \\ Y_{3..} \\ Y_{.1.} \\ Y_{.2.} \\ Y_{.3.} \\ Y_{..A} \\ Y_{..B} \\ Y_{..C} \end{bmatrix}$$

If we add the independent equations $\sum_{h=1}^{3} \rho_h = \sum_{i=1}^{3} \gamma_i = \sum_{i=A}^{c} \tau_j = 0$, we should have full rank if the design is connected, and we obtain the following:

$$
\left[\begin{array}{c|ccc|ccc|ccc}
7 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
0 & 2 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 2
\end{array}\right]
\begin{bmatrix}
\hat{\mu} \\ \hat{\rho}_1 \\ \hat{\rho}_2 \\ \hat{\rho}_3 \\ \hat{\gamma}_1 \\ \hat{\gamma}_2 \\ \hat{\gamma}_3 \\ \hat{\tau}_A \\ \hat{\tau}_B \\ \hat{\tau}_C
\end{bmatrix}
=
\left[\begin{array}{ll}
Y_{...} & = \text{grand total} \\
\left.\begin{array}{l} Y_{1..} \\ Y_{2..} \\ Y_{3..} \end{array}\right\} & = \text{row totals} \\
\left.\begin{array}{l} Y_{.1.} \\ Y_{.2.} \\ Y_{.3.} \end{array}\right\} & = \text{column totals} \\
\left.\begin{array}{l} Y_{..A} \\ Y_{..B} \\ Y_{..C} \end{array}\right\} & = \text{treatment totals}
\end{array}\right]
$$

From the above we note that

$$\hat{\mu} = \frac{7}{6}\,\bar{y}_{...} - \frac{1}{6}\left(\bar{y}_{3..} + \bar{y}_{.3.} + \bar{y}_{..B}\right)$$

$$\hat{\rho}_3 = \bar{y}_{3..} - \hat{\mu}$$

$$\hat{\gamma}_3 = \bar{y}_{.3.} - \hat{\mu}$$

$$\hat{\tau}_B = \bar{y}_{..B} - \hat{\mu}$$

However we cannot obtain solutions for the remaining unknowns unless either $\hat{\rho}_1 = \hat{\rho}_2 = 0$, $\hat{\gamma}_1 = \hat{\gamma}_2 = 0$, or $\hat{\tau}_A = \hat{\tau}_C = 0$. This is illustrated below for the six equations involving these unknowns:

$$\begin{bmatrix} 2 & 0 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \hat{\gamma}_1 \\ \hat{\gamma}_2 \\ \hat{\tau}_A \\ \hat{\tau}_C \end{bmatrix} = \begin{bmatrix} Y_{1..} \\ Y_{2..} \\ 2\,Y_{.1.} + Y_{1..} \\ 2\,Y_{.2.} + Y_{2..} \\ 2\,Y_{..A} + 2\,Y_{1..} + 2\,Y_{.1.} \\ 2\,Y_{..C} + 2\,Y_{2..} + 2\,Y_{.2.} \end{bmatrix}$$

For the six equations the last two are linear combinations of the first four equations.

It is conjectured that for any latin square of order k for which k-1 entries are missing such that k-1 different rows, k-1 different columns, and k-1 different treatments are involved will be in the class of designs furnishing counter examples to the above statement of row-column-treatment connectedness. It should be noted that if the two missing observations in the 3 X 3 latin square are for the same treatment, row, or column, the design will be connected.